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Non-perturbative states in type II superstring theory from classical spinning membranes

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Abstract

We find a new family of exact solutions in membrane theory, representing toroidal membranes spinning in several planes. They have energy square proportional to the sum of the different angular momenta, generalizing Regge-type string solutions to membrane theory. By compactifying the eleven dimensional theory on a circle and on a torus, we identify a family of new non-perturbative states of type IIA and type IIB superstring theory (which contains the perturbative spinning string solutions of type II string theory as a particular case). The solution represents a spinning bound state of D branes and fundamental strings. Then we find similar solutions for membranes on $AdS_7 \times S^4$ and $AdS_4 \times S^7$. We also consider the analogous solutions in $SU(N)$ matrix theory, and compute the energy. They can be interpreted as rotating open strings with D0 branes attached to their endpoints.

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1 Introduction

Understanding new aspects of M-theory may eventually lead to a powerful setup to uncover and clarify the non-perturbative physics of string theory (for a review of M-theory, see [1]). An important piece of information about the theory is its mass spectrum. In eleven uncompact dimensions, determining the spectrum of quantum states is a hard problem, there is no coupling constant and gravitational effects are always important. In compactifying the eleven dimensional coordinate X^{10} on a circle of radius R_{10}, one makes contact with type IIA superstring theory with coupling $g_{\rm IIA} = 2\pi R_{10}^3/l_P^3$, where l_P is the eleven-dimensional Planck length [2, 3]. For small radius, M-theory describes a weakly-coupled string theory and it is possible to compute masses without the complication of gravitational interactions, provided they are of order $M = O(g_{\rm IIA}^{-s})$, $s < 2$ so that the gravitational forces (proportional to $g_{\rm IIA}^2 M$) are negligible as $g_{\rm IIA} \to 0$. In particular, this is the reason why there exists a simple weak coupling description for D0 branes, which have masses $O(g_{\rm IIA}^{-1})$.

M-theory is known to contain membranes and five branes (see e.g. [1, 4]). In this paper we will find new classical solutions in supermembrane theory [5, 6] and compute their energy. In flat space, these solutions are similar to the spinning string solutions of [7], which were found to be highly stable quantum mechanically in weakly-coupled string theory. For a special choice of quantum numbers (such that the solution depends only on one spatial membrane coordinate) the solution reduces to the string solution of [7]. For general quantum numbers, the energy has non-trivial dependence on the coupling constant $g_{\rm IIA}$ of the form $E \sim O(g_{\rm IIA}^{-1})$. This means that

for small radius R_{10} one can ignore gravitational effects, so these solutions describe genuine quantum states with high-excitation quantum numbers in M-theory, and new non-perturbative states in type IIA string theory. By further compactifying the coordinate X^9 on a circle of radius R_9, one makes contact with type IIB string theory. Ten-dimensional type IIB theory arises as M-theory on the torus (X^9, X^{10}) in the limit that the area $R_9 R_{10}$ goes to zero at fixed $g_{\rm IIB} = \frac{R_{10}}{R_9}$. Membranes states can lead to various types of non-perturbative objects of type IIB theory, such as bound states of fundamental strings and D strings [8–10]. The spinning solutions presented here lead to a new class of non-perturbative states of type II string theory, representing rotating objects which do not have a pure fundamental string interpretation. They have in general D brane charges and fundamental string charges.

We then consider similar spinning solutions for membrane theory in $AdS_7 \times S^4$ and in $AdS_4 \times S^7$. These solutions are the membrane analogue of the general class of spinning string solutions found in $AdS_5 \times S^5$ in [11] (which includes some spinning string solutions appeared earlier in [12, 13]). The energy admits a simple expansion at large angular momenta.

It has been proposed [14] that M-theory in the light-cone frame is described by Matrix theory, which can be viewed as a regularized theory of the supersymmetric membrane [15] (for reviews see [16, 17]). The fundamental degrees of freedom can be viewed as the D0 branes, which in the light-cone frame are expected to capture all the complicated dynamics of M theory. The Lagrangian is that of supersymmetric quantum mechanics described by 0+1 super Yang-Mills theory with sixteen supersymmetries. In the last part of the paper we consider "spinning" solutions in Matrix theory, found in [18], which are analogous to the spinning membrane solutions. Here they are slightly generalized to incorporate rotation in four different planes, and in addition we compute the energy, which was not done in [18].

There is an extensive literature on classical solutions in membrane/matrix theory. The reader can look at [6, 19] and more recently [18, 20–25]. Some of the present solutions are similar to the ones obtained in [18]. In addition, we present generalizations and many new solutions that include momentum and winding in the directions X_9, X_{10}, which allows to make contact with type II string theories. These are in fact the most interesting solutions that give rise to new non-perturbative states with rotation in type II string theory.

2 Spinning membranes solutions in flat space

2.1 General rotating ansatz

The starting point is the bosonic part of the action for the supermembrane [5] in flat space

$$S = -\frac{T_2}{2} \int d^3\xi \left(\sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \sqrt{-h} \right) \ , \quad \mu = 0, \ldots, 10 \ . \tag{2.1}$$

T_2 is the membrane tension, $T_2 = (2\pi l_P^3)^{-1}$. The equations of motion for the metric on the membrane worldvolume gives

$$h_{\alpha\beta} = \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta} \eta_{\mu\nu}, \qquad \alpha, \beta = 0, 1, 2 \ . \tag{2.2}$$

In what follows we use the notation $(\xi^0, \xi^1, \xi^2) \equiv (\tau, \sigma, \rho)$. Treating the X and h as independent fields, we now fix the gauge in the usual way [6, 26], by setting

$$h_{\tau\sigma} = h_{\tau\rho} = 0 \ , \quad h_{\tau\tau} L^2 = -g \ , \tag{2.3}$$

where L is an arbitrary constant with units of length and g the determinant of the spacelike part of the induced metric. The action (2.1) becomes

$$S = -\frac{T_2}{2}\int d^3\xi \left(-L\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + \frac{g}{L}\left(h^{ij}\partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} - 1\right)\right) \ . \tag{2.4}$$

Using the equation of motion for the induced metric, $h_{ij} = \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu}$ the action (2.4) reads

$$S = -\frac{T_2}{2}\int d^3\xi \left(-L\partial_\tau X^\mu \partial_\tau X_\mu + \frac{1}{2L}\{X_\mu, X_\nu\}\{X^\mu, X^\nu\}\right) \ , \tag{2.5}$$

where the Poisson bracket is defined as usual as

$$\{f, g\} \equiv \epsilon^{rs}\partial_r f \partial_s g \ , \qquad r, s = \sigma, \rho \ , \tag{2.6}$$

for any two differentiable functions f, g on the two dimensional manifold.

In terms of the X^μ fields, the constraints for the induced metric (2.3) that must be imposed to the solutions are

$$\eta_{\mu\nu}\frac{\partial X^\mu}{\partial\tau}\frac{\partial X^\nu}{\partial\sigma} = \eta_{\mu\nu}\frac{\partial X^\mu}{\partial\tau}\frac{\partial X^\nu}{\partial\rho} = 0 \ , \tag{2.7}$$

$$L^2\eta_{\mu\nu}\frac{\partial X^\mu}{\partial\tau}\frac{\partial X^\nu}{\partial\tau} = \left(\eta_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma}\frac{\partial X^\nu}{\partial\rho}\right)^2 - \left(\eta_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma}\frac{\partial X^\nu}{\partial\sigma}\right)\left(\eta_{\mu\nu}\frac{\partial X^\mu}{\partial\rho}\frac{\partial X^\nu}{\partial\rho}\right) \ . \tag{2.8}$$

Now we consider a general rotating ansatz of the following form

$$\begin{aligned}
X_0 &= \kappa\tau \ , \\
Z_1(\tau,\sigma,\rho) &= X_1 + iX_2 = r_1(\sigma,\rho)e^{i\omega_1\tau + i\alpha_1(\sigma,\rho)} \ , \\
Z_2(\tau,\sigma,\rho) &= X_3 + iX_4 = r_2(\sigma,\rho)e^{i\omega_2\tau + i\alpha_2(\sigma,\rho)} \ , \\
Z_3(\tau,\sigma,\rho) &= X_5 + iX_6 = r_3(\sigma,\rho)e^{i\omega_3\tau + i\alpha_3(\sigma,\rho)} \ , \\
Z_4(\tau,\sigma,\rho) &= X_7 + iX_8 = r_4(\sigma,\rho)e^{i\omega_4\tau + i\alpha_4(\sigma,\rho)} \ , \\
X_9 &= X_{10} = 0 \ .
\end{aligned} \tag{2.9}$$

This represents a membrane spinning in four orthogonal planes $Z_1, \ldots, Z_4$. Using this ansatz, the action for the membrane (2.5) reads

$$S = -\frac{T_2}{2}\int d^3\xi \left(L(\kappa^2 - \sum_{a=1}^{4}\omega_a^2 r_a^2) + \frac{1}{2L}\sum_{a,b=1}^{4}\left(\{r_a, r_b\}^2 + r_a^2 r_b^2\{\alpha_a, \alpha_b\}^2 + 2r_a^2\{\alpha_a, r_b\}^2\right)\right) \ . \tag{2.10}$$

The equations of motion for the radial and the angular coordinates are given by

$$r_c\left(L^2\omega_c^2 - \sum_{a=1}^{4}(r_a^2\{\alpha_c, \alpha_a\}^2 + \{r_a, \alpha_c\}^2)\right) + \sum_{a=1}^{4}\left(\{\{r_c, r_a\}, r_a\} + r_a^2\{\{r_c, \alpha_a\}, \alpha_a\}\right) = 0 \tag{2.11}$$

$$\sum_{a=1}^{4}\left(r_a^2\{\{\alpha_c, \alpha_a\}, \alpha_a\} + \{\{\alpha_c, r_a\}, r_a\}\right) = 0 \ . \tag{2.12}$$

Inserting the ansatz (2.9), the constraints (2.7), (2.8) take the form

$$\sum_{a=1}^{4} \omega_a r_a^2 \partial_\sigma \alpha_a = \sum_{a=1}^{4} \omega_a r_a^2 \partial_\rho \alpha_a = 0 \ , \tag{2.13}$$

$$L^2(\kappa^2 - \sum_{a=1}^{4} \omega_a^2 r_a^2) = \sum_{a,b=1}^{4} \left(\frac{1}{2}\{r_a, r_b\}^2 + \frac{1}{2} r_a^2 r_b^2 \{\alpha_a, \alpha_b\}^2 + r_a^2 \{\alpha_a, r_b\}^2 \right) \ . \tag{2.14}$$

2.2 Constant radius solution

A simplification of our original ansatz (that generalizes the rotating string solutions of [7] to membrane theory) is considering solutions of constant radius and with the phases α_a depending linearly on $\sigma,\ \rho$. Assuming that the r_a are constants, the equation of motion (2.12) for the phases becomes

$$\sum_{a=1}^{4} r_a^2 \{\{\alpha_b, \alpha_a\}, \alpha_a\} = 0 \ . \tag{2.15}$$

This is indeed solved by $\alpha_a(\sigma, \rho) = k_a \sigma + l_a \rho$, i.e.,

$$Z_a(\tau, \sigma, \rho) = r_a e^{i\omega_a \tau + i(k_a \sigma + l_a \rho)} \ , \quad a = 1, \ldots, 4 \ , \tag{2.16}$$

with $k_a,\ l_a$ integer numbers, as implied by the periodicity condition satisfied by a closed membrane. The equation of motion (2.11) of the radial coordinates leads to

$$L\omega_b^2 - \frac{1}{L} \sum_{a=1}^{4} r_a^2 (l_b k_a - l_a k_b)^2 = 0 \ . \tag{2.17}$$

This equation determines the frequencies ω_b as a function of the radii r_a and the different winding numbers.

Using the ansatz (2.16), the constraints can be written as

$$\sum_{a=1}^{4} \omega_a r_a^2 k_a = \sum_{a=1}^{4} \omega_a r_a^2 l_a = 0 \ , \tag{2.18}$$

$$L^2(\kappa^2 - \sum_{a=1}^{4} \omega_a^2 r_a^2) = \sum_{a,b=1}^{4} r_a^2 r_b^2 (k_a l_b - k_b l_a)^2 \ . \tag{2.19}$$

Combining with eq. (2.17), we get

$$\kappa^2 = 2 \sum_{a=1}^{4} r_a^2 \omega_a^2 \ . \tag{2.20}$$

The solution (2.16) appeared in [18] in the case of light-cone gauge membrane theory. In the light-cone gauge the constraints (2.18) (which lead to constraints on the angular momenta and winding numbers, see (2.27)) is absent. The reason is the following. This constraint comes from

(2.7). In the light cone gauge $X^+ = p_+\tau$, this gives $p_+\partial_s X^- = \partial_\tau X^i \partial_s X^i$, which can be solved for X^- provided the integrability condition $\{\dot{X}^i, X^i\} = 0$ is satisfied, which is automatically the case for the solution (2.16). In the present case, instead, $X_0 = \kappa\tau$, $X_9 = X_{10} = 0$, and this physical membrane state (2.16) exists only if the angular momenta and winding numbers are constrained by (2.18) (i.e. (2.27)). The light-cone gauge represents an extreme physical situation of infinite momentum, where the theory typically gets simplified. In section 2.4 we shall consider a generalization where the constraint (2.7) can be solved by adding winding and linear momentum in the directions $X_9,\ X_{10}$.

2.3 Energy and angular momenta

Now we evaluate the conserved quantum numbers of the rotating membrane solutions. We start with the action (2.5),

$$S = -\frac{T_2}{2}\int d^3\xi \Bigg(L(\dot{X_0}^2 - \dot{r_5}^2 - \dot{r_6}^2 - \sum_{a=1}^{4}(r_a^2\dot{\beta_a}^2 + \dot{r}_a^2\beta_a^2)) \\ + \frac{1}{L}\{r_a, X_0\}^2 + \frac{1}{L}\sum_{a,b=1}^{6}\left(\frac{1}{2}\{r_a, r_b\}^2 + \frac{1}{2}(r_a r_b)^2\{\beta_a,\beta_b\}^2 + r_b^2\{r_a,\beta_b\}^2\right)\Bigg) , \tag{2.21}$$

with coordinates defined as

$$\begin{aligned} &Z_a(\tau,\sigma,\rho) = r_a(\tau,\sigma,\rho)e^{i\beta_a(\tau,\sigma,\rho)}, \quad a = 1,...,4 \\ &X_9 \equiv r_5\ , \quad X_{10} \equiv r_6\ , \quad X_0\ . \end{aligned} \tag{2.22}$$

The conserved quantum numbers are

$$E = \int d\sigma d\rho\ \Pi_0, \quad \Pi_0 \equiv \frac{\delta S}{\delta \dot{X}^0} \tag{2.23}$$

$$J_a = \int d\sigma d\rho\ \Pi_{\beta_a}, \quad \Pi_{\beta_a} \equiv \frac{\delta S}{\delta \dot{\beta}^a} \tag{2.24}$$

In the present case, we obtain

$$E = 4\pi^2 T_2 L\ \kappa\ , \qquad J_a = 4\pi^2 T_2 L\ \omega_a r_a^2\ . \tag{2.25}$$

The energy of our solution can be obtained from (2.20) (coming from the constraint (2.8)), which determines κ,

$$E^2 = 2(4\pi^2 T_2 L)\sum_a J_a\omega_a\ , \tag{2.26}$$

where ω_a are determined by (2.17). Note that each term in the sum on the right hand side of (2.26) is positive definite, since $J_a\omega_a = 4\pi^2 T_2 L\ \omega_a^2 r_a^2$. The constraints (2.18) become

$$\sum_{a=1}^{4} J_a k_a = 0\ , \qquad \sum_{a=1}^{4} J_a l_a = 0\ . \tag{2.27}$$

Let us now particular cases where some of the angular momenta vanish. Because of the constraint (2.18), the minimum number of non-vanishing angular momenta is two. However, in

the case of rotation in two planes, i.e, $Z_3, Z_4 = const$, it can be seen that eqs. (2.17), (2.18) lead to $\omega_1 = \omega_2 = 0$, so this solution is trivial. The first nontrivial case is the rotation in three planes.[1]

In the case of rotation in three planes, we can write (2.17), (2.18) as

$$\sum_{a=1}^{3} J_a k_a = 0 \ , \qquad \sum_{a=1}^{3} J_a l_a = 0 \ , \tag{2.28}$$

$$\tilde{\omega}_b^2 - \sum_{a=1}^{3} \frac{J_a}{\tilde{\omega}_a} (l_b k_a - l_a k_b)^2 = 0 \ , \tag{2.29}$$

where we have defined $\tilde{\omega}_a \equiv (4\pi^2 T_2 L^3)^{1/3} \omega_a$. In (2.29) we have expressed the radii in terms of the angular momenta using (2.25). The reason is that the energy must be expressed in terms of the conserved quantum numbers, angular momenta and winding numbers, characterizing the state (winding numbers are not conserved in interactions because they are along contractible circles; nevertheless, they characterize the states of the free theory). The parameters $\tilde{\omega}_a$ are dimensionless (just as ω_a), but they depend only on J_a, l_a, k_a through (2.29). Equations (2.29) are coupled cubic equations, and the analytic solution is complicated. It is easy to see that solutions with real frequencies (and real radii) exist. For example, with the choice

$$(k_1, k_2, k_3) = (1, -3, -4) \ , \quad (l_1, l_2, l_3) = (4, 3, -11) \ , \quad (J_1, J_2, J_3) = (4, 2, 2) \ , \tag{2.30}$$

the constraints (2.28) are satisfied, and from (2.29) we obtain the real solution $(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3) = (4.9, 21.5, 14.4)$ (for which also the radii are real, since J_a/ω_a are positive, see (2.25)). In conclusion, the energy in the case of rotation in three planes is

$$E^2 = 2(4\pi^2 T_2)^{2/3} \sum_{a=1}^{3} J_a \tilde{\omega}_a \ , \tag{2.31}$$

where the $\tilde{\omega}_a$ are determined by eq. (2.29) in terms of integer quantum numbers k_a, l_a, J_a obeying (2.28). As expected, the energy does not depend on the arbitrary length parameter L introduced in the choice of gauge.

2.4 More general rotating solutions

We now consider the membrane wrapped around two compact directions, X^9 and X^{10}, with linear momentum,

$$\begin{aligned} X_9 &= R_9 (n_9 \sigma + m_9 \rho) + q_9 \tau \ , \\ X_{10} &= R_{10} (n_{10} \sigma + m_{10} \rho) + q_{10} \tau \ . \end{aligned} \tag{2.32}$$

[1] When the coordinates X_9, X_{10} are compact, there are solutions with rotation in one and two planes, described in section 2.4. In the case of eleven uncompact coordinates, there is a solution with rotation in two planes in the light-cone gauge, where the constraint (2.18) is absent. In section 5 we discuss the analogous solution in the Matrix theory.

For the coordinates Z_a, $a = 1, ..., 4$, we use the same ansatz (2.16). The constraints (2.7), (2.8) now take the form

$$\sum_{a=1}^{4} \omega_a r_a^2 k_a + R_9 q_9 n_9 + R_{10} q_{10} n_{10} = 0 \ , \tag{2.33}$$

$$\sum_{a=1}^{4} \omega_a r_a^2 l_a + R_9 q_9 m_9 + R_{10} q_{10} m_{10} = 0 \ , \tag{2.34}$$

$$L^2(\kappa^2 - \sum_a \omega_a^2 r_a^2 - q_9^2 - q_{10}^2) = L^2 \sum_{a=1}^{d} r_a^2 \omega_a^2 + R_9^2 R_{10}^2 (n_9 m_{10} - n_{10} m_9)^2 \ , \tag{2.35}$$

where in the last equation we have used the relation

$$\tilde{\omega}_c^2 - \sum_a \frac{J_a^2}{\tilde{\omega}_a} (l_c k_a - l_a k_c)^2 - (4\pi^2 T_2)^{2/3} R_9^2 (m_9 k_c - n_9 l_c)^2 - (4\pi^2 T_2)^{2/3} R_{10}^2 (m_{10} k_c - n_{10} l_c)^2 = 0, \tag{2.36}$$

which follows from the equation of motion for the radial coordinates. The parameters $\tilde{\omega}_a$ are defined as before, $\tilde{\omega}_a \equiv (4\pi^2 T_2 L^3)^{1/3} \omega_a$.

The linear momenta along X_9 and X_{10} are

$$P_i = \int d\sigma d\rho \ \Pi_i, \quad \Pi_i \equiv \frac{\delta S}{\delta \dot{X}^i} \ . \tag{2.37}$$

We obtain

$$P_9 = 4\pi^2 T_2 L q_9 \ , \qquad P_{10} = 4\pi^2 T_2 L q_{10} \ . \tag{2.38}$$

Because X_9 and X_{10} are compact coordinates, they are quantized,

$$P_9 \equiv \frac{\tilde{n}_9}{R_9} \ , \quad P_{10} \equiv \frac{\tilde{n}_{10}}{R_{10}} \ . \tag{2.39}$$

The constraints take the form

$$\sum_{a=1}^{4} J_a k_a + n_9 \tilde{n}_9 + n_{10} \tilde{n}_{10} = 0 \ , \qquad \sum_{a=1}^{4} J_a l_a + m_9 \tilde{n}_9 + m_{10} \tilde{n}_{10} = 0 \ . \tag{2.40}$$

The energy is then obtained from (2.25) with κ determined by (2.35). We get

$$E^2 = 2(4\pi^2 T_2)^{2/3} \sum_{a=1}^{4} J_a \tilde{\omega}_a + (4\pi^2 T_2)^2 R_9^2 R_{10}^2 (n_9 m_{10} - n_{10} m_9)^2 + \frac{\tilde{n}_9^2}{R_9^2} + \frac{\tilde{n}_{10}^2}{R_{10}^2} \ . \tag{2.41}$$

The $\tilde{\omega}_a$ are numbers determined by (2.36) in terms of the integer quantum numbers J_a, l_c, k_c, m_9, n_9, m_{10}, n_{10} and the parameters R_9, R_{10}. Note that in the energy E there is no dependence on the arbitrary constant L as expected. The second term proportional to $R_9^2 R_{10}^2$ is the usual contribution to the energy coming from the torus area, times membrane tension, times membrane charge (equal to $n_9 m_{10} - n_{10} m_9$).

2.4.1 Rotation in one plane

The simplest rotating solution is the case Z_2, Z_3, Z_4 =const. corresponding to rotation in one plane Z_1, with the previous ansatz for X_9, X_{10},

$$
\begin{aligned}
X_0 &= \kappa\tau \ , \\
Z_1 &= re^{i\omega\tau + i(k\sigma + l\rho)} \ , \\
X_9 &= R_9(n_9\sigma + m_9\rho) + q_9\tau \ , \\
X_{10} &= R_{10}(n_{10}\sigma + m_{10}\rho) + q_{10}\tau \ .
\end{aligned}
$$

In this case, the constraints become,

$$
\begin{aligned}
&kJ + n_9\tilde{n}_9 + n_{10}\tilde{n}_{10} = 0 \ , \\
&lJ + m_9\tilde{n}_9 + m_{10}\tilde{n}_{10} = 0 \ , \\
&L^2\omega^2 = R_9^2(m_9k - n_9l)^2 + R_{10}^2(m_{10}k - n_{10}l)^2 \ .
\end{aligned}
$$

The energy is then

$$
\begin{aligned}
E^2 &= 2(4\pi^2T_2L)J\omega + (4\pi^2T_2)^2R_9^2R_{10}^2(n_9m_{10} - n_{10}m_9)^2 + \frac{\tilde{n}_9^2}{R_9^2} + \frac{\tilde{n}_{10}^2}{R_{10}^2} \\
&= \left((4\pi^2T_2)(n_9m_{10} - n_{10}m_9)R_9R_{10} + \frac{1}{R_9}\sqrt{\tilde{n}_9^2 + \tilde{n}_{10}^2\frac{R_9^2}{R_{10}^2}} \right)^2 \ . \qquad (2.42)
\end{aligned}
$$

Thus the energy is a complete square. This is related to the fact that the corresponding quantum state is a BPS state. In fact, it is the same BPS state studied in [8,9]. In general, the BPS state of [8, 9] represents a non-marginal bound state of fundamental string and D string in type IIB string theory, with charges $\tilde{n}_9$, $\tilde{n}_{10}$ and momentum $n \equiv n_9m_{10} - n_{10}m_9$ (see also section 3.2). It describes an excited string state (or excited membrane state), whose specific quantum state can be any of the exponential number of states at this level. The present solution represents one of these quantum states, namely the state with maximum angular momentum.

2.4.2 Rotation in two planes

Let us now consider the solution describing rotation in two planes Z_1, Z_2, by setting $Z_3 = Z_4$ =const. The constraints and the equations of motion become,

$$
\begin{aligned}
&k_1J_1 + k_2J_2 + n_9\tilde{n}_9 + n_{10}\tilde{n}_{10} = 0 \ , \\
&l_1J_1 + l_2J_2 + m_9\tilde{n}_9 + m_{10}\tilde{n}_{10} = 0 \ , \\
&\tilde{\omega}_1^2 = \frac{J_2}{\tilde{\omega}_2}(k_1l_2 - k_2l_1)^2 + (4\pi^2T_2)^{2/3}\left[R_9^2(m_9k_1 - n_9l_1)^2 + R_{10}^2(m_{10}k_1 - n_{10}l_1)^2\right] \ , \\
&\tilde{\omega}_2^2 = \frac{J_1}{\tilde{\omega}_1}(k_1l_2 - k_2l_1)^2 + (4\pi^2T_2)^{2/3}\left[R_9^2(m_9k_2 - n_9l_2)^2 + R_{10}^2(m_{10}k_2 - n_{10}l_2)^2\right] \ .
\end{aligned}
$$

The solution simplifies when the linear momenta along the directions coordinates X_9, X_{10} vanish, i.e. $\tilde{n}_9 = \tilde{n}_{10} = 0$. In this case, we have

$$
J_1k_1 + J_2k_2 = J_1l_1 + J_2l_2 = 0 \ , \qquad (2.43)
$$

so that $l_1 k_2 = l_2 k_1$, and we find an explicit simple expression for the frequencies

$$\tilde{\omega}_1^2 = (4\pi^2 T_2)^{2/3} \Big[R_9^2 (m_9 k_1 - n_9 l_1)^2 + R_{10}^2 (m_{10} k_1 - n_{10} l_1)^2 \Big] \tag{2.44}$$

$$\tilde{\omega}_2^2 = (4\pi^2 T_2)^{2/3} \Big[R_9^2 (m_9 k_2 - n_9 l_2)^2 + R_{10}^2 (m_{10} k_2 - n_{10} l_2)^2 \Big] \ . \tag{2.45}$$

Then, the energy of this solution is

$$\begin{aligned} E^2 =& 2(4\pi^2 T_2)^{2/3} \sum_{a=1}^{2} J_a \tilde{\omega}_a + (4\pi^2 T_2)^2 R_9^2 R_{10}^2 (n_9 m_{10} - n_{10} m_9)^2 \\ =& 2(4\pi^2 T_2) \sqrt{R_9^2 \left(\frac{k_1}{l_1} m_9 - n_9 \right)^2 + R_{10}^2 \left(\frac{k_1}{l_1} m_{10} - n_{10} \right)^2} \, (|l_1 J_1| + |l_2 J_2|) \\ &+ (4\pi^2 T_2)^2 R_9^2 R_{10}^2 (n_9 m_{10} - n_{10} m_9)^2 \end{aligned} \tag{2.46}$$

We have added absolute value bars to the two terms in the sum to account for the fact that each term in the sum $\sum_a J_a \tilde{\omega}_a$ is positive definite because $J_a \sim r_a^2 \tilde{\omega}_a$. Note that, unlike the previous case, the energy is not a complete square.

In the case $X_9 = n_9 R_9 \sigma$ and $X_{10} = R_{10} \rho$, i.e. $m_9 = n_{10} = 0,\ m_{10} = 1$, the energy becomes

$$E^2 = 2(4\pi^2 T_2) \sqrt{R_9^2 n_9^2 + R_{10}^2 \frac{k_1^2}{l_1^2}} \, (|l_1 J_1| + |l_2 J_2|) + (4\pi^2 T_2)^2 R_9^2 R_{10}^2 n_9^2 \ . \tag{2.47}$$

We will return to this energy formula in the next section.

3 Non-perturbative states in type II string theory from rotating membranes

3.1 Type IIA string theory or M-theory on S^1

Type IIA string theory is obtained from M-theory by compactifying the eleventh dimension X^{10} on a circle. The string tension and string coupling are related to the membrane and M-theory parameters as follows:

$$\alpha' = \frac{1}{4\pi^2 R_{10} T_2}, \qquad g_{\rm IIA}^2 = 4\pi^2 R_{10}^3 T_2 \ , \tag{3.1}$$

$$T_1 = (2\pi\alpha')^{-1} = 2\pi R_{10} T_2 \ , \tag{3.2}$$

$$\kappa_{11}^2 = 16\pi^5 l_P^9 \ , \quad T_2 = (2\pi l_P^3)^{-1} = (4\pi^2 \alpha'^{3/2} g_{\rm IIA})^{-1} \ , \quad R_{10}^2 = \alpha' g_{\rm IIA}^2 \ . \tag{3.3}$$

The perturbative solutions of type IIA string theory have the form dictated by the "double dimensional reduction" ansatz [27],

$$X^{10} = R_{10} \rho \qquad \partial_\rho X^\mu = 0, \quad \mu = 0, \dots, 9 \ . \tag{3.4}$$

The membrane solutions of the previous sections have explicit dependence on the ρ coordinate, and also momentum in the eleventh coordinate X^{10}. One expects that in string theory they arise as non-perturbative objects.

First, consider the type IIA limit in ten uncompact dimensions which is achieved by setting $R_9 \to \infty$. In this case we must set $n_9 = m_9 = 0$ in the solution (2.16), (2.32). Now the momentum P_9 is continuous. In addition, we set $n_{10} = 0$, $m_{10} = 1$ to have $X^{10} = R_{10}\rho + q_{10}\tau$. In terms of the type IIA parameters, the energy (2.41) of the membrane solution of section 2.4 then takes the form

$$E^2 = P_9^2 + \frac{2}{\alpha' g_{\rm IIA}^{2/3}} \sum_{a=1}^{4} J_a\, \tilde{\omega}_a(g_{\rm IIA}) + \frac{\tilde{n}_{10}^2}{\alpha' g_{\rm IIA}^2}\ , \tag{3.5}$$

where angular momenta and winding numbers satisfy the constraints

$$\sum_{a=1}^{4} k_a J_a = 0\ , \qquad \sum_{a=1}^{4} l_a J_a = -\tilde{n}_{10}\ . \tag{3.6}$$

The $\tilde{\omega}_a$ are determined in terms of the coupling constant $g_{\rm IIA}$ and the conserved quantum numbers by the system of equations (2.36), which in terms of string parameters reads

$$\tilde{\omega}_c^2 - \sum_a \frac{J_a^2}{\tilde{\omega}_a}(l_c k_a - l_a k_c)^2 - g_{\rm IIA}^{4/3} k_c^2 = 0\ . \tag{3.7}$$

In order to solve explicitly the above equations, we consider again the case of rotation in two planes. Then the equations for the frequencies are given by

$$\begin{aligned} \tilde{\omega}_1^2 - \frac{J_2^2}{\tilde{\omega}_2}(l_1 k_2 - k_1 l_2)^2 &= g_{\rm IIA}^{4/3} k_1^2\ , \\ \tilde{\omega}_2^2 - \frac{J_1^2}{\tilde{\omega}_1}(l_1 k_2 - k_1 l_2)^2 &= g_{\rm IIA}^{4/3} k_2^2\ . \end{aligned} \tag{3.8}$$

These equations admit a number of solutions, which simplify in the particular case of spins of equal magnitude and opposite sign, $J_1 = -J_2$ (which, by the constraint equations (3.6), implies $k_1 = k_2$). In this case the frequencies are given by

$$\tilde{\omega}_1 = \frac{2 g_{\rm IIA}^{8/3} k_1^2}{\tilde{n}_{10}^2} \frac{1}{1 + \sqrt{1 + \frac{4 g_{\rm IIA}^4 k_1^2}{\tilde{n}_{10}^4}}}\ , \tag{3.9}$$

$$\tilde{\omega}_2 = -\frac{\tilde{n}_{10}^2}{2 g_{\rm IIA}^{4/3}} \left(1 + \sqrt{1 + \frac{4 g_{\rm IIA}^4 k_1^2}{\tilde{n}_{10}^4}} \right)\ . \tag{3.10}$$

Inserting into (3.5), we obtain the exact expression for the energy of this configuration. At weak coupling, $g_{\rm IIA} \ll 1$, the energy has the following expansion

$$\alpha' E^2 = \frac{2 J_1 \tilde{n}_{10}^2}{g_{\rm IIA}^2} + 4 J_1 g_{\rm IIA}^2 \frac{k_1^2}{\tilde{n}_{10}^2} + \mathcal{O}\left(g_{\rm IIA}^6\right)\ . \tag{3.11}$$

There are several points which are worth noticing. First, we again find a Regge-type formula. A priori, this is not obvious from (3.5), since the frequencies in general depend on the angular momenta. The point is that the frequencies depend on the combination $J_{1,2}^2(l_1k_2 - l_2k_1)^2$ which for $J_1 = -J_2$ is equal to $\tilde{n}_{10}^2$ (see (3.6)). Second, also for this solution the energy has the behavior $E = \text{const.}\frac{1}{g_{\text{IIA}}}$, characteristic of D branes. As explained in the introduction, this guarantees that one can ignore gravitational back reaction effects in the weak coupling limit. The origin of the $1/g_{\text{IIA}}$ behavior is a D0 brane charge, coming from momentum of the membrane in the X_{10} direction. Because the membrane has zero charge, now there is no D2 brane and no winding charge for the fundamental string. Thus the state represents a rotating system of D0 brane and fundamental string in uncompactified ten-dimensional type IIA string theory.

Now we consider the case of compact X_9 coordinate, and the solution with energy given by (2.47). In terms of type IIA string theory parameters, the analytical energy formula (2.47) –describing rotation in two planes– becomes

$$E^2 = \frac{2}{\alpha'}\sqrt{\frac{R_9^2 n_9^2}{\alpha' g_{\text{IIA}}^2} + \frac{k_1^2}{l_1^2}}\ (|l_1J_1| + |l_2J_2|) + \frac{R_9^2 n_9^2}{\alpha'^2}\ , \tag{3.12}$$

$$k_1J_1 + k_2J_2 = l_1J_1 + l_2J_2 = 0\ .$$

At weak coupling, the energy behaves as $E = \text{const.}\frac{1}{g_{\text{IIA}}}$. This behavior is characteristic of D branes. For this particular solution there is no D0 brane present, since we have set the D0 brane charge $\tilde{n}_{10}$ to zero. The origin of the $1/g_{\text{IIA}}$ behavior is the presence of a D2 brane. Indeed, for this solution, the membrane charge is equal to $n_9m_{10} - n_{10}m_9 = n_9$. Since the membrane is also extended in the Z_1, Z_2 and X_{10} directions, after dimensional reduction one is left with a rotating bound state of a D2 brane and a fundamental string (with winding charge also equal to n_9). Remarkably, the energy formula has a simple Regge-type behavior, $E^2 \sim J$.

Now consider the particular case $n_9 = 0$. We get

$$E^2 = 2(4\pi^2 T_2 R_{10})\,(|k_1J_1| + |k_2J_2|) = \frac{2}{\alpha'}\,(|k_1J_1| + |k_2J_2|) \tag{3.13}$$

It is interesting that the energy of these states does not depend on l_1, l_2. In particular, the energy is the same as in the case $l_1 = l_2 = 0$, which in the dimensionally reduced theory corresponds to a string (see (2.16), (3.4)). Thus there is a family of membranes (parametrized by l_1, with $l_2 = -l_1J_1/J_2$) giving rise to states with the same energy in the string theory.

The solutions with $l_1 = l_2 = 0$ and energy given by (3.13) are the solutions of [7]. These solutions were found to be quantum mechanically very stable, with a lifetime proportional to [7] $g_{\text{IIA}}^{-2}(\text{mass})^5$. The reason of this long life time is that the closed string cannot classically break due to the fact that during the evolution there is never contact between two points of the string. It can only decay by emitting light modes. These states are probably the most stable states in the string spectrum. We expect that the present membrane solutions with $n_9 \neq 0$ and energy given by (3.12) are also long-lived quantum mechanically.

An interesting question is what is the supergravity solution describing these spinning membranes. Spinning M-brane supergravity solutions were given in [28]. However, none of these solutions describe the present configurations. The reason is that the present membrane configurations rotate along the directions where the membrane is extended.

3.2 Type IIB string theory

Type IIB string theory arises by compactifying M theory on the torus X_9, X_{10}. The type IIB parameters are related to M theory parameters as follows:

$$\alpha' = \frac{1}{4\pi^2 R_{10} T_2} , \qquad g_{\rm IIB}^2 = \frac{R_{10}^2}{R_9^2} , \qquad R_{9B} = \frac{\alpha'}{R_9} . \tag{3.14}$$

In terms of type IIB parameters, the general expression of the energy (2.41) reads

$$E^2 = 2\left(\frac{R_{9B}}{g_{\rm IIB}\alpha'^2}\right)^{2/3} \sum_{a=1}^{4} J_a \tilde{\omega}_a + \frac{1}{R_{9B}^2}(n_9 m_{10} - n_{10} m_9)^2 + \frac{R_{9B}^2}{\alpha'^2}\left(\tilde{n}_9^2 + \frac{\tilde{n}_{10}^2}{g_{\rm IIB}^2}\right) . \tag{3.15}$$

Let us now consider particular cases. In the case of rotation on one plane, eq. (2.42), we have

$$E^2 = \left((n_9 m_{10} - n_{10} m_9)\frac{1}{R_{9B}} + \frac{R_{9B}}{\alpha'}\sqrt{\tilde{n}_9^2 + \frac{\tilde{n}_{10}^2}{g_{\rm IIB}^2}}\right)^2 , \tag{3.16}$$

describing the BPS non-marginal bound state of D string (charge $\tilde{n}_{10}$), fundamental string (charge $\tilde{n}_9$) and momentum $n \equiv n_9 m_{10} - n_{10} m_9$, as anticipated in section 2.4.1.

In the case of the solution (2.47) describing rotation in two planes, we have

$$E^2 = \frac{2}{\alpha'}\sqrt{\frac{n_9^2}{g_{\rm IIB}^2} + \frac{k_1^2}{l_1^2}}\,\left(|l_1 J_1| + |l_2 J_2|\right) + \frac{n_9^2}{R_{9B}^2} . \tag{3.17}$$

This is the T-dual of the D2 brane/fundamental string system described in section 3.1. Because the D2 brane is extended in X_9 and Z_1, Z_2 directions, T-duality in X_9 should produce a complicated rotating system involving D-strings, D3 branes and fundamental strings, with momentum n_9. It is remarkable that the energy of such a system is given by a simple Regge-type formula. As the type IIA T-dual counterpart, we expect that this system is classically stable and long-lived quantum mechanically.

4 Spinning membranes in $AdS_p \times S^q$

In this section we construct analogous membrane solutions in a curved background spacetime, $AdS_p \times S^q$, which are relevant for AdS/CFT correspondence applications [29]. First we obtain the general expression for our solutions, which are of the form (2.16), and then we find an explicit expression for the energy of these solutions in the large angular momentum limit. The solutions generalize the rotating circular strings of [11] to membrane theory.

We consider membranes on $AdS_7 \times S^4$ and $AdS_4 \times S^7$ or, generically, $AdS_p \times S^q$. Let $Y^\mu, \mu = 0, \ldots, p$ be the embedding coordinates in the AdS space and $X^k, k = 1, \ldots, q+1$ the embedding coordinates in the sphere. The action for the membrane reads

$$S = \frac{T_2}{2}\int d^3\xi \Bigg(- \sqrt{-h}h^{\alpha\beta}\left(\partial_\alpha Y^\mu \partial_\beta Y^\nu \eta_{\mu\nu} + \partial_\alpha X^k \partial_\beta X^k\right) + \frac{1}{2}\sqrt{-h}$$
$$+ \tilde{\Lambda}\left(Y^\mu Y^\nu \eta_{\mu\nu} + R_A^2\right) + \Lambda\left(X^k X^k - R_S^2\right)\Bigg) . \tag{4.1}$$

The Lagrange multipliers enforce the constraints

$$\sum_{k=1}^{q+1} X_k^2 = R_S^2, \qquad Y^\mu Y^\nu \eta_{\mu\nu} = -R_A^2 \ , \tag{4.2}$$

where $\eta_{\mu\nu} = (-,+,\ldots,+,-)$. We have defined $\tilde{\Lambda}$, Λ so that they transform as $\sqrt{-h}$ under world-volume reparametrizations. The radii of the sphere and of the AdS space are given by

$$\begin{aligned} R_S = 2R_{AdS} = l_P(2^5\pi^2 N)^{1/2}, \quad (AdS_4 \times S^7) \ , \\ R_S = R_{AdS}/2 = l_P(\pi N)^{1/3}, \quad (AdS_7 \times S^4) \ , \end{aligned} \tag{4.3}$$

where l_P is the Planck length. From the action (4.1) one derives the following equations of motion for the coordinates,

$$\Lambda X_k = \partial_\beta \left(\sqrt{-h} h^{\alpha\beta} \partial_\alpha X_k \right) \ , \tag{4.4}$$

$$\tilde{\Lambda} Y_\mu = \partial_\beta \left(\sqrt{-h} h^{\alpha\beta} \partial_\alpha Y_\mu \right) \ . \tag{4.5}$$

Now we consider the following ansatz,

$$\begin{aligned} Z_1 \equiv X_1 + iX_2 = r_1 e^{i\omega_1\tau + im_1\sigma + in_1\rho} \ , \\ Z_2 \equiv X_3 + iX_4 = r_2 e^{i\omega_2\tau + im_2\sigma + in_2\rho} \ , \\ \vdots \\ Z_d \equiv X_{2d-1} + iX_{2d} = r_d e^{i\omega_d\tau + im_d\sigma + in_d\rho} \ , \end{aligned} \tag{4.6}$$

where $d = 2$ for S^4 and $d = 4$ for S^7. For the AdS coordinates, we take

$$Z_0 \equiv Y_0 + iY_p = R_A e^{i\omega_0\tau} \ . \tag{4.7}$$

We use the same gauge as in section 2, given by $h_{\tau\sigma} = 0$, $h_{\tau\rho} = 0$ and $h_{\tau\tau}L^2 = -g$. For our ansatz (4.6), these constraints read

$$\sum_{a=1}^{d} m_a \omega_a r_a^2 = \sum_{a=1}^{d} n_a \omega_a r_a^2 = 0 \ , \tag{4.8}$$

$$R_A^2 \omega_0^2 - \sum_{a=1}^{d} r_a^2 \omega_a^2 = \frac{1}{L^2} \sum_{b<a} r_a^2 r_b^2 (m_b n_a - m_a n_b)^2 \ . \tag{4.9}$$

The equations of motion that are derived from the action (4.1) give rise to the following relations between the parameters:

$$\tilde{\Lambda} = -L\omega_0^2 \ , \tag{4.10}$$

$$\begin{aligned} -\Lambda &= L\omega_a^2 - \frac{1}{L} h_{\rho\rho} m_a^2 - \frac{1}{L} h_{\sigma\sigma} n_a^2 + \frac{2}{L} h_{\sigma\rho} m_a n_a \\ &= Lw_a^2 - \frac{1}{L} \sum_{b=1}^{d} r_b^2 \left(n_b m_a - n_a m_b \right)^2 \ . \end{aligned} \tag{4.11}$$

The energy and the angular momenta can be derived from (2.23), (2.24),

$$J_a = 4\pi^2 T_2 L r_a^2 \omega_a \equiv 4\pi^2 T_2 L \; R_S^2 \mathcal{J}_a \ , \tag{4.12}$$

$$E = 4\pi^2 T_2 L R_A \omega_0 \equiv 4\pi^2 T_2 L R_A \; \mathcal{E} \ . \tag{4.13}$$

The constraint $\sum_{a=1}^d r_a^2 = R_S^2$ can be written as

$$\sum_{a=1}^{d} \frac{\mathcal{J}_a}{w_a} = R_S^2 \ . \tag{4.14}$$

Solving (4.11), one finds the frequencies in terms of Λ. Then Λ is determined using (4.14). Plugging the result into (4.9) (with $r_a^2 = R_S^2 \mathcal{J}_a/\omega_a$), one determines the energy (4.13) in terms of J_a and winding numbers. The system of equations can be solved systematically as power series in $\frac{1}{\mathcal{J}}$,

$$\mathcal{J} \equiv \sum_{a=1}^{d} \mathcal{J}_a \ . \tag{4.15}$$

Note that large $\mathcal{J}$ implies also large Lagrange multiplier $|\Lambda|$. Indeed, since the radii r_a on the sphere are bounded, large angular momenta requires large frequencies, which, by (4.11), implies large $|\Lambda|$, and the scaling $|\Lambda| \sim \mathcal{J}^2$. This situation is similar to that of spinning strings, see [11]. So we first solve equation (4.11) for the frequencies (inserting $r_a^2 = R_S^2 \mathcal{J}_a/\omega_a$) using perturbation theory and then plug the expression of the frequencies in the constraint (4.14) to express Λ in terms of the angular momenta and the winding numbers. We find

$$w_c = \mathcal{J} + \frac{R_S^2}{2L^2 \mathcal{J}^2} \sum_{a=1}^{d} \mathcal{J}_a (n_c m_a - n_a m_c)^2 + \mathcal{O}\left(\frac{1}{\mathcal{J}^2}\right) \ , \tag{4.16}$$

$$|\Lambda|^{1/2} = \mathcal{J} - \frac{R_S^2}{2L^2 \mathcal{J}^3} \sum_{a,c}^{d} \mathcal{J}_a \mathcal{J}_c (n_c m_a - n_a m_c)^2 + \mathcal{O}\left(\frac{1}{\mathcal{J}^2}\right) \ . \tag{4.17}$$

Hence we find the energy:

$$\mathcal{E}^2 = \omega_0^2 = \frac{R_S^2}{R_A^2}\mathcal{J}^2 + \frac{R_S^4}{\mathcal{J}^2 R_A^2 L^2} \sum_{a,b=1}^{d} (m_a n_b - n_a m_b)^2 \, \mathcal{J}_a \mathcal{J}_b + \mathcal{O}\left(\frac{1}{\mathcal{J}^2}\right) \ . \tag{4.18}$$

Taking the square root and expanding, we get

$$\mathcal{E} - \frac{R_S}{R_A}\mathcal{J} = \frac{R_S^3}{2\mathcal{J}^3 R_A L^2} \sum_{a,b=1}^{d} (m_a n_b - n_a m_b)^2 \mathcal{J}_a \mathcal{J}_b + \mathcal{O}\left(\frac{1}{\mathcal{J}^2}\right) \ . \tag{4.19}$$

Finally, expressing eq. (4.19) in terms of the physical energy and spin, one obtains that the dependence on the arbitrary constant L cancels, as expected, to give the result,

$$E - \frac{J}{R_S} = \frac{R_S^5 (4\pi^2 T_2)^2}{2J^3} \sum_{a,b=1}^{d} (m_a n_b - n_a m_b)^2 J_a J_b + \mathcal{O}\left(\frac{1}{J^2}\right) \ , \tag{4.20}$$

supplemented with the constraints (4.8)

$$\sum_{a=1}^{d} J_a m_a = \sum_{a=1}^{d} J_a n_a = 0 \ . \tag{4.21}$$

This spinning toroidal membrane solution is the analog of the circular spinning strings in $AdS_5 \times S^5$ found in [11]. For the spinning strings, in the large angular momentum limit, one finds the formula

$$\mathcal{E} - \mathcal{J} = \frac{1}{2\mathcal{J}^2} \sum_{i=1}^{3} m_i^2 \mathcal{J}_i + \ldots \ . \tag{4.22}$$

As in the case of $AdS_5 \times S^5$ (4.22), the energy (4.20) of configurations with angular momentum on the sphere exhibits the behavior $E \sim J$, as opposed to the Regge behavior $E^2 \sim J$ of flat space. In the framework of the AdS/CFT correspondence, the energy formula (4.20) gives the anomalous dimension of a CFT operator with $SO(5)$ (for the 5+1 CFT) or $SO(8)$ for the 2+1 CFT) charges J_a.

In the particular case of rotation in two planes, there is a significant simplification, because in this case the constraint (4.8) implies that $n_1 m_2 - m_2 n_1 = 0$. As a result, the energy is given by the leading order term without higher order corrections,

$$E = \frac{J}{R_S} \ , \tag{4.23}$$

This suggests that this configuration should be supersymmetric and that the corresponding CFT operator should be BPS, since its bare dimension is exact in the classical membrane approximation.

5 Membranes and Matrix theory

In this section we consider solutions of the matrix model equations which are the analogues of the classical membrane solutions of Section 2.2, and study their properties.

5.1 Membranes solutions and matrix model solutions

The starting point is the bosonic part of the supersymmetric quantum-mechanics Yang-Mills Lagrangian describing the dynamics of N D0 branes, given by

$$L = \frac{1}{2R_{10}} \left(\mathrm{tr} \left[\dot{X}^i \dot{X}^i + \frac{R_{10}^2}{2l_P^6} \left[X^i, X^j \right]^2 \right] \right) , \quad i = 1, \ldots, 8 \ , \tag{5.1}$$

where X^i are matrices of the Lie algebra of $SU(N)$. The equations of motion are

$$\frac{d^2}{dt^2} X^k = -\frac{R_{10}^2}{l_P^6} \sum_{i=1}^{8} \left[\left[X^k, X^i \right], X^i \right] \ . \tag{5.2}$$

This equation of motion is supplemented with the self-consistency Gauss constraint,

$$\sum_{i=1}^{8} \left[X_i, \dot{X}_i \right] = 0 \ . \tag{5.3}$$

In order to establish a dictionary between membrane solutions and matrix model solutions, let us recall how the toroidal membrane arises as the $N = \infty$ limit of the $SU(N)$ matrix model. The toroidal membrane coordinates can be expanded as

$$X^i\,(\tau,\sigma,\rho) = \sum_{n,m=-\infty}^{\infty} X^i_{nm}(\tau) e^{in\sigma + im\rho} \equiv \sum_{n,m=-\infty}^{\infty} X^i_{nm}(\tau) T_{nm}\ , \tag{5.4}$$

where $T_{n_1 n_2} \equiv T_{\vec{n}}$ are generators of the area-preserving diffeomorphism algebra of the torus,

$$\{T_{\vec{n}}, T_{\vec{m}}\} = (\vec{n}\times\vec{m}) T_{\vec{n}+\vec{m}}\ , \qquad (\vec{n}\times\vec{m}) \equiv n_1 m_2 - m_1 m_2\ . \tag{5.5}$$

The analogue of the $T_{\vec{n}}$ generators in the $SU(N)$ matrix model is a special basis of generators $\{J_{\vec{n}}\}$ of the Lie algebra of $SU(N)$ satisfying the algebra [30, 31]

$$[J_{\vec{m}}, J_{\vec{n}}] = -2i \sin\left(\frac{2\pi}{N}(\vec{n}\times\vec{m})\right) J_{\vec{m}+\vec{n}}\ , \qquad N = \text{odd}\ , \tag{5.6}$$

$$[J_{\vec{m}}, J_{\vec{n}}] = -2i \sin\left(\frac{\pi}{N}(\vec{n}\times\vec{m})\right) J_{\vec{m}+\vec{n}}\ , \qquad N = \text{even}\ . \tag{5.7}$$

In the Appendix we review the construction and properties of these algebras. A general matrix on the Lie algebra can be expanded as

$$X^i\,(\tau) = \sum_{n,m=0}^{N-1}{}'\ X^i_{nm}(\tau)\ J_{nm}\ , \tag{5.8}$$

where prime means excluding the term $(n, m) = (0, 0)$. In the $N = \infty$ limit, the algebra (5.6), (5.7) approaches the area-preserving diffeomorphism algebra of the torus, and the $SU(N)$ matrix theory is expected to reproduce exactly the same dynamics of membrane theory in the light-cone gauge, since they are described by the same Hamiltonian.

Given any toroidal membrane classical solution, it can be expanded as in (5.4). Then one can write down an ansatz for a matrix model solution of the form (5.8) by taking the same coefficients X^i_{nm} and extending the sum taking into account the periodicity, $J_{(m_1+Nk_1, m_2+Nl_1)} = J_{(m_1,m_2)}$. For the rotating membrane solutions of section 2, the ansatz is straightforward since in complex coordinates Z_a there is a single term in the sum (5.4), i.e. $Z_a = r_a e^{i\omega_a\tau}\ e^{ik_a\sigma + il_a\rho}$.

5.2 Rotating matrix model solution

Here we review the rotating solution of the matrix model given in [18]. The starting point is the matrix model Lagrangian (5.1) in complexified coordinates as

$$L = \frac{1}{2R_{10}}\mathrm{tr}\left[\dot{Z}_a \dot{Z}_a^\dagger\right] + \frac{R_{10}}{8l_P^6}\mathrm{tr}\Bigg(\ [Z_a, Z_b]\left[Z_a^\dagger, Z_b^\dagger\right] + \left[Z_a, Z_b^\dagger\right]\left[Z_a^\dagger, Z_b\right]\ \Bigg)\ , \qquad a, b = 1, \dots, 4\ . \tag{5.9}$$

The equations of motion of the matrix model and the constraint take the form

$$\frac{d^2}{dt^2} Z_a = -\frac{R_{10}^2}{2l_P^6}\Bigg(\ \left[Z_b^\dagger, [Z_b, Z_a]\right] + \left[Z_b, \left[Z_b^\dagger, Z_a\right]\right]\ \Bigg)\ , \tag{5.10}$$

$$\left[\dot{Z}_a, Z_a^\dagger\right] + \left[\dot{Z}_a^\dagger, Z_a\right] = 0 \ . \tag{5.11}$$

A solution with rotation in the planes Z_a can now be found by analogy with the membrane case, i.e. replacing $T_{nm} = e^{in\sigma + im\rho}$ by J_{nm}, with the matrix ansatz

$$Z_a = r_a e^{i\omega_a t} J_{\vec{n}_a} \ , \quad a = 1, \ldots, d \ , \tag{5.12}$$

where $1 < d \leq 4$. Consider the case of even N. The equations of motion (5.10) give the relation

$$\omega_a^2 = 4\frac{R_{10}^2}{l_P^6} \sum_{b=1}^{d} r_b^2 \sin^2\left(\frac{\pi}{N}\left(\vec{n}_b \times \vec{n}_a\right)\right) \ , \tag{5.13}$$

which is the analog of the relation (2.17). The Gauss constraint (5.11) is satisfied automatically by the ansatz (5.12), using that $J_{\vec{m}}^\dagger = J_{-\vec{m}}$.

5.3 Energy of this configuration

Let us now evaluate the light-cone energy P_- of the solutions that we have found above. This is obtained by inserting the solution (5.12) into the Hamiltonian corresponding to the matrix model (5.9). We obtain

$$P_- = H = \frac{N}{2R_{10}} \sum_{a=1}^{d} \omega_a^2 r_a^2 - \frac{NR_{10}}{l_P^6} \sum_{a,b=1}^{d} r_a^2 r_b^2 \sin^2\left(\frac{\pi}{N}\vec{n}_a \times \vec{n}_b\right) \ . \tag{5.14}$$

Using eq. (5.13) for the frequencies, the energy can be expressed in a compact form as

$$P_- = \frac{NR_{10}}{l_P^6} \sum_{a,b=1}^{d} r_a^2 r_b^2 \sin^2\left(\frac{\pi}{N}\vec{n}_a \times \vec{n}_b\right) \ . \tag{5.15}$$

For large N, this has the same form as the membrane energy (2.26), which (using (2.17), (2.25)) can be written as

$$E_{\rm mem}^2 = {\rm const.} \sum_{a,b=1}^{d} r_a^2 r_b^2 (l_b k_a - l_a k_b)^2 \ .$$

Let us now consider explicit cases. In the case of $SU(2)$, the simplest non-trivial example, the generators of the Lie algebra (5.7) coincide with the usual Pauli matrices, as can be seen in the Appendix. For this group we find that any non-trivial configuration (with non-zero energy) has an energy given by

$$P_- = \frac{2R_{10}}{l_P^6} \sum_{a,b=1, a\neq b}^{d} r_a^2 r_b^2 \sin^2\left(\frac{\pi}{2}\right) = \frac{2R_{10}}{l_P^6} \sum_{a,b=1, a\neq b}^{d} r_a^2 r_b^2 \ . \tag{5.16}$$

In a similar way one can obtain the energy for the case of $SU(3)$, but now it can be seen that there are configurations with different energy.

The matrix model solution can be viewed pictorially as follows. Recall that the diagonal entries in the matrices Z_a represent strings which begin and end on the same D0 brane, whereas

the non-diagonal entries represent strings going from one D0 brane to a different one. For the $SU(3)$ case, there are three D0 branes. The Z_a can be proportional to any of the eight generators listed in appendix A. Solutions with a Z_a proportional to a generator with three non-diagonal entries represent a configuration of three strings joining the three D0 branes, forming a triangle, which rotate in the plane Z_a. In another plane Z_b, the solution can have Z_b proportional to a diagonal generator $J_{(1,0)}$ or $J_{(2,0)}$, representing a rotating system of three D0 branes with strings beginning and ending on the same D0 brane.

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Appendix A Construction of $SU(N)$ generators

In this appendix we review the construction and relevant properties of the $SU(N)$ generators in the representation that we are using [30]. First we do this in general and then we give the explicit expressions for the $SU(3)$ generators. These algebra becomes the area-preserving diffeomorphism algebra of the toroidal membrane in the $N \to \infty$ limit.

A basis for the $SU(N)$ algebras can be built from two unitary $N \times N$ matrices,

$$g \equiv \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ 0 & 0 & \omega^2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \omega^{N-1} \end{pmatrix}, \quad h \equiv \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad g^N = h^N = 1 \ , \tag{A.1}$$

where ω is an N'th root of unity with period no smaller than N, that is, $\omega = e^{i4\pi/N}$ for N odd and $\omega = e^{i2\pi/N}$ for N even. Now using $hg = \omega gh$, it follows that the unitary $N \times N$ matrices

$$J_{\vec{m}} = \omega^{\frac{m_1 m_2}{2}} g^{m_1} h^{m_2}, \quad \vec{m} \equiv (m_1, m_2) \ , \tag{A.2}$$

span the algebra of $SU(N)$, that is, they close under multiplication

$$J_{\vec{n}} J_{\vec{m}} = \omega^{-\frac{(\vec{n}\times\vec{m})}{2}} J_{\vec{n}+\vec{m}} \ , \tag{A.3}$$

with ω as defined above, so that their commutation rules are given in terms of trigonometric structure constants,

$$[J_{\vec{m}}, J_{\vec{n}}] = \left(\omega^{\vec{n}\times\vec{m}/2} - \omega^{-\vec{n}\times\vec{m}/2}\right) J_{\vec{m}+\vec{n}} = -2i \sin\left(\frac{2\pi}{N} \vec{m} \times \vec{n}\right) J_{\vec{m}+\vec{n}} \ , \tag{A.4}$$

where the last equality follows for odd N (for even N one has π instead of 2π), and the vector product is defined in the usual way, $\vec{n} \times \vec{m} = n_1 m_2 - m_1 n_2$. Another interesting property is that for the case of three generators whose winding number satisfy

$$\vec{n}_1 + \vec{n}_2 + \vec{n}_3 = 0 \ , \tag{A.5}$$

then the algebra of these generators gets simplified and we get a twisted $SU(2)$ trigonometric algebra

$$[J_{\vec{n}_1}, J_{\vec{n}_2}] = -2i \sin\left(\frac{2\pi}{N}\left(\vec{n_1} \times \vec{n_2}\right)\right) J^{\dagger}_{\vec{n}_3}\ , \tag{A.6}$$

together with the corresponding permutations. This algebra is twisted $SU(2)$ in the sense that it is the algebra of $SU(2)$ but with a single trigonometric structure constants, since due to the condition (A.5) it follows that $\vec{n_1} \times \vec{n_2} = \vec{n_1} \times \vec{n_3} = \vec{n_2} \times \vec{n_3}$.

From their definition, one can check that all the generators are traceless. Consider, in particular, the generators of the form

$$J_{(m_1,0)}\ , \quad m_1 < N \tag{A.7}$$

The trace is given by

$$\mathrm{tr}\left(J_{m_1,0}\right) = \mathrm{tr}\ g^{m_1} = \sum_{k=0}^{N-1} \omega^k\ , \tag{A.8}$$

which indeed vanishes, since ω is the $N-$th root of unity. These generators are normalized as follows,

$$\mathrm{tr}\left(J_{\vec{n}} J^{\dagger}_{\vec{n}}\right) = \mathrm{tr}\left(J_{\vec{n}} J_{-\vec{n}}\right) = \mathrm{tr}\ 1 = N\ . \tag{A.9}$$

Now we compute the explicit expressions for the generators of the $SU(2)$ and $SU(3)$ algebra in the basis of [30]. For $SU(2)$ the generators reduce to the usual Pauli matrices,

$$J_{(1,0)} = \sigma_3\ , \quad J_{(0,1)} = \sigma_1\ , \quad J_{(1,1)} = -\sigma_2\ . \tag{A.10}$$

For $SU(3)$ the expressions are more involved, differing from the usual canonical basis. These generators of the algebra of $SU(3)$ are given by

$$J_{(1,0)} = \begin{pmatrix} 1 & & \\ & e^{i4\pi/3} & \\ & & e^{i2\pi/3} \end{pmatrix}, \quad J_{(2,0)} = \begin{pmatrix} 1 & & \\ & e^{i2\pi/3} & \\ & & e^{i4\pi/3} \end{pmatrix}$$

$$J_{(0,1)} = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}, \quad J_{(0,2)} = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}$$

$$J_{(1,1)} = \begin{pmatrix} & e^{i2\pi/3} & \\ & & 1 \\ e^{i4\pi/3} & & \end{pmatrix}, \quad J_{(2,2)} = \begin{pmatrix} & & e^{i2\pi/3} \\ e^{i4\pi/3} & & \\ & 1 & \end{pmatrix} \tag{A.11}$$

$$J_{(2,1)} = \begin{pmatrix} & e^{i4\pi/3} & \\ & & 1 \\ e^{i4\pi/3} & & \end{pmatrix}, \quad J_{(2,2)} = \begin{pmatrix} & & e^{i4\pi/3} \\ e^{i2\pi/3} & & \\ & 1 & \end{pmatrix} \tag{A.12}$$

One can explicitly check with this equations the algebra of the generators, given by

$$[J_{\vec{m}}, J_{\vec{n}}] = -2i \sin\left(\frac{2\pi}{3}\vec{m} \times \vec{n}\right) J_{\vec{m}+\vec{n}} \tag{A.13}$$

subject to the periodicity of the generators of the algebra, namely

$$J_{(m_1+3k_1,m_2+3l_1)} = J_{(m_1,m_2)} \tag{A.14}$$

with k_a, l_a integer numbers. One can explicitely check that all these generators are indeed traceless, that is

$$\mathrm{tr}\,(J_{\vec{n}}) = 0\ . \tag{A.15}$$

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